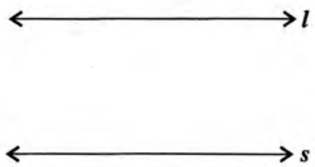
16 Loci

(Locus and Its Constructions)

16.1 **Locus** is a Latin word from which the words *location, locality*, etc., are derived.

16.2 Definition :

Locus is the path traced by a moving point, which moves so as to satisfy the certain given condition/conditions.



1 Two parallel lines *l* and *s* are 4 cm apart. Find the locus of a point which is always equidistant from both the given lines.

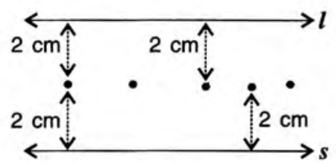
Solution :

Condition : The moving point is always equidistant from the given parallel lines *l* and *s*.

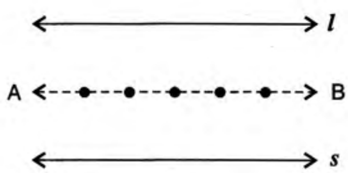
(i) As the distance between the given parallel lines is 4 cm and moving point is equidistant from these lines, so mark some points each $\frac{4}{2} = 2$ cm from *l* and *s*. [see fig. (i)]

(ii) On joining all the points marked, a straight line AB is obtained which is the required locus. [see fig. (ii)]

(i)



(ii)



From the final figure obtained, the required locus is the line AB which is parallel to both the given lines *l* and *s* and is also equidistant from both the lines.

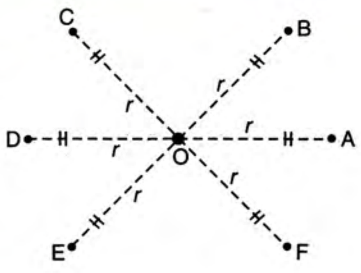
The plural of locus is **loci** (pronounced as **losai**)

2 Show that the locus of a point equidistant from a fixed point is a circle with the fixed point as centre.

Solution :

Let O be the fixed point and we have to find the locus of a moving point P which moves in such a way that the distance between the moving point P and the fixed point O is always the same.

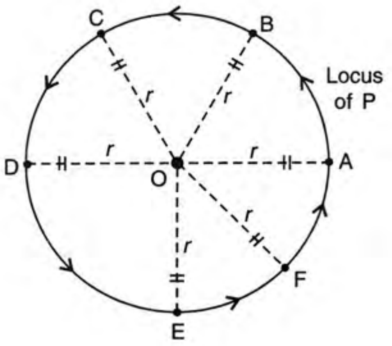
If the distance between the moving point P and the fixed point O is *r* cm, mark some points A, B, C, D, E, ..., *etc*, each at a distance of *r* cm from the fixed point O.



Now draw a free-hand curve through the marked points A, B, C, D, ..., *etc.*

We shall find that the final figure obtained is a circle with fixed point as centre and the distance between the moving point and the fixed point as radius.

Thus, **the locus of a point equidistant from a fixed point is a circle with the fixed point as centre**.



16.3 Theorems Based on Symmetry :

Theorem 3

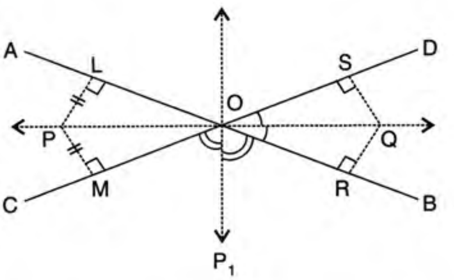
The locus of a point equidistant from two intersecting lines is the bisector of the angles between the lines.

Given : Two straight lines AB and CD intersecting at O. A point P is the interior of angle AOC such that it is equidistant from AB and CD.

To Prove : Locus of P is the bisector of angle AOC.

i.e. (i) P lies on bisector of angle AOC, and *conversely.*

(ii) every other point on the bisector of $\angle$AOC is equidistant from the intersecting lines AB and CD.



Construction : Draw a line through O and P. Then draw PL perpendicular to AB and PM perpendicular to CD.

(i) Proof :

Statement	***Reason***
In triangles POL and POM :	
1. PL = PM	P is equidistant from AB and CD [Given]
2. $\angle$PLO = $\angle$PMO	Each is 90° [By construction]
3. PO = PO	Common
$\therefore$ Δ POL $\cong$ Δ POM	R.H.S.
$\therefore$ $\angle$POL = $\angle$POM	Corresponding parts of congruent triangles are congruent

Therefore, **P lies on the bisector of angle AOC.**

(ii) Conversely : Let Q be any point on the bisector OP.

Now to show that Q is equidistant from AB and CD, draw QR and QS perpendiculars to AB and CD respectively.

Clearly, Δ OQR $\cong$ Δ OQS [By A.A.S. or A.S.A.]

$\Rightarrow$ QR = QS [C.P.C.T.C.]

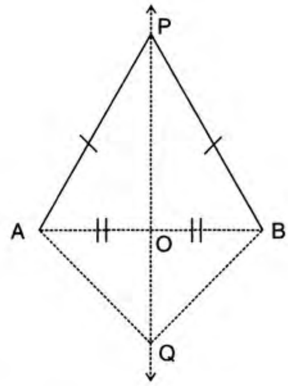
$\Rightarrow$ **Q is equidistant from AB and CD**

The same results can be proved by taking a point, in the interior of angle COB or in the interior of angle AOD, etc.

Hence the theorem is proved.

Theorem 4

The locus of a point equidistant from two given points is the perpendicular bisector of the line joining the two points.



Given : Two fixed points A and B. P is a moving point equidistant from A and B, *i.e.* PA = PB.

To Prove : Locus of moving point P is the perpendicular bisector of line AB.

i.e. (i) P lies on the perpendicular bisector of AB and *conversely;*

(ii) every other point on this perpendicular bisector is equidistant from A and B.

Construction : Join AB. Locate 'O' the mid-point of AB. Join P and O.

(i) Proof :

Statement	***Reason***
In triangles AOP and BOP :	
1. PA = PB	Given
2. AO = BO	'O' is the mid-point of AB
3. PO = PO	Common
$\therefore \Delta AOP \cong \Delta BOP$	S.S.S.
$\therefore \angle AOP = \angle BOP = 90°$	Since, $\angle AOP + \angle BOP = 180°$.

Therefore, **P lies on the perpendicular bisector of AB.**

(ii) *Conversely* : Let Q be any other point on line PO, the perpendicular bisector of AB.

Clearly, $\Delta AOQ \cong \Delta BOQ$ [By S.A.S.]

$\therefore$ AQ = BQ [Corresponding parts of congruent triangles are congruent]

$\Rightarrow$ **Every point on the perpendicular bisector is equidistant from A and B.**

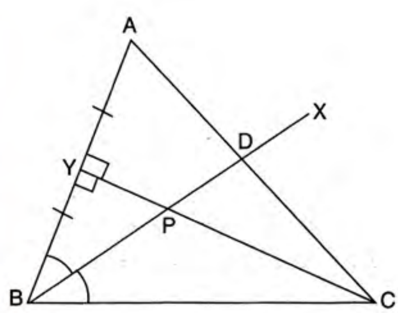
Hence the theorem is proved.

16.4 Applications :

3 In the given figure, BX bisects angle ABC and intersects AC at point D. Line segment CY is perpendicular to AB and intersects BX at point P. If Y is mid-point of AB, prove that :

(i) point P is equidistant from A and B.

(ii) point D is equidistant from AB and BC.



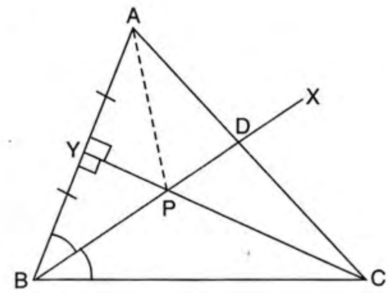
Solution :

(i) Join A and P.

In Δ APY and Δ BPY,

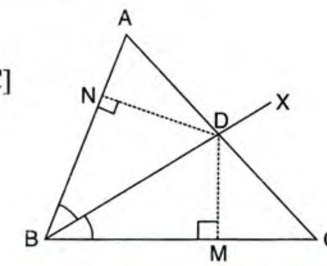
1. AY = BY [Given, Y is mid-point of AB]
2. PY = PY [Common]
3. ∠AYP = ∠BYP [Each 90°]
∴ Δ APY ≅ Δ BPY [S.A.S.]
⇒ AP = BP [C.P.C.T.C.]
⇒ **P is equidistant from A and B**

Hence Proved.



(ii) Draw DM and DN perpendiculars to sides BC and BA respectively.
In Δ DBM and Δ DBN,
1. BD = BD [Common]
2. ∠DBM = ∠DBN [Given, BX bisects ∠ABC]
3. ∠DMB = ∠DNB [Each 90°]
∴ Δ DBM ≅ Δ DBN [A.A.S.]
⇒ DM = DN
⇒ **point D is equidistant from AB and BC.**

Hence Proved.

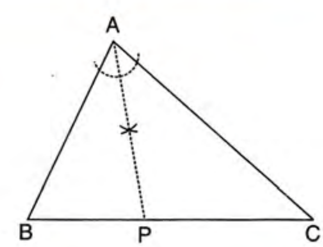


4 Find a point on the base of a scalene triangle equidistant from its sides.

Solution :

Let the given triangle be Δ ABC, whose base is BC.

Draw the bisector of angle of vertex BAC, which meets base BC at P. ***P is the required point.***



Reason :

Since, each point on the angle bisector is equidistant from the arms of the angle and P lies on the bisector of ∠BAC.

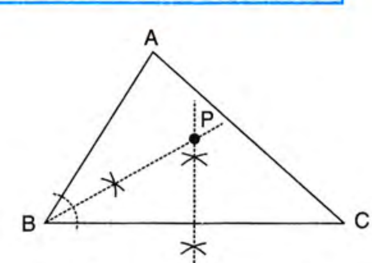
∴ **P is equidistant from AB and AC.**

5 Construct a triangle ABC in which AB = 6 cm, BC = 7 cm and CA = 6·5 cm. Find a point P equidistant from B and C; and also equidistant from AB and BC.

Solution :

From the given data, construct triangle ABC. Draw bisector of angle ABC and perpendicular bisector of BC; both intersecting at P.

P is the required point.



Reason :

Since; (i) P is on perpendicular bisector of BC, P is equidistant from B and C;
(ii) P is on bisector of angle ABC, P is equidistant from AB and BC.

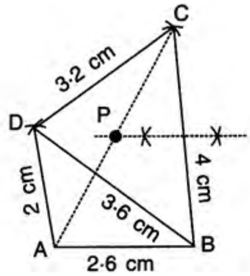
6 Construct a quadrilateral ABCD, having given AB = 2·6 cm; BC = 4·0 cm, CD = 3·2 cm, AD = 2 cm and diagonal BD = 3·6 cm.
Mark a point P on diagonal AC, equidistant from B and C.

Solution :

Obtain the quadrilateral ABCD, by first constructing triangle ABD and then triangle BCD.

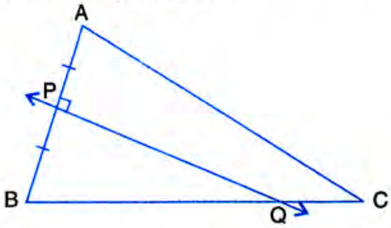
Draw perpendicular bisector of BC which meets AC at P.

Thus, P is the required point on diagonal AC, equidistant from B and C.



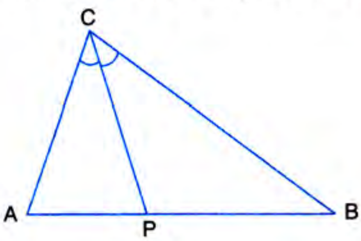
EXERCISE 16(A)

1. Given : PQ is perpendicular bisector of side AB of the triangle ABC.



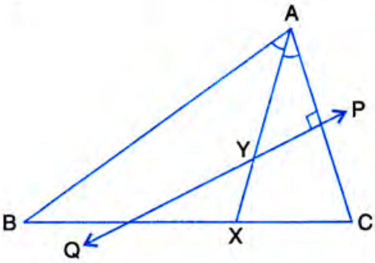
Prove : Q is equidistant from A and B.

2. Given : CP is bisector of angle C of Δ ABC.



Prove : P is equidistant from AC and BC.

3. Given : AX bisects angle BAC and PQ is perpendicular bisector of AC which meets AX at point Y.



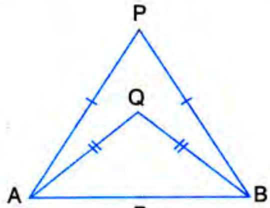
Prove : (i) X is equidistant from AB and AC.

(ii) Y is equidistant from A and C.

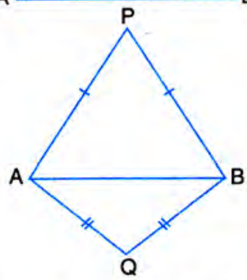
4. Construct a triangle ABC, in which AB = 4·2 cm, BC = 6·3 cm and AC = 5 cm. Draw perpendicular bisector of BC which meets AC at point D. Prove that D is equidistant from B and C.

5. In each of the given figures; PA = PB and QA = QB.

(i)



(ii)



Prove, in each case, that PQ (produce, if required) is perpendicular bisector of AB.

Hence, state the locus of the points equidistant from two given fixed points.

6. Construct a right angled triangle PQR, in which ∠Q = 90°, hypotenuse PR = 8 cm and QR = 4·5 cm. Draw bisector of angle PQR and let it meet PR at point T. Prove that T is equidistant from PQ and QR.

7. Construct a triangle ABC in which angle ABC = 75°, AB = 5 cm and BC = 6·4 cm. Draw perpendicular bisector of side BC and also the bisector of angle ACB. If these bisectors intersect each other at point P; prove that P is equidistant from B and C; and also from AC and BC.

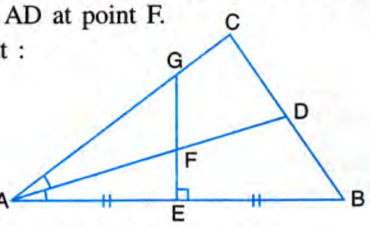
8. In parallelogram ABCD, side AB is greater than side BC and P is a point in AC such that PB bisects angle B.

Prove that P is equidistant from AB and BC.

9. In triangle LMN, bisectors of interior angles at L and N intersect each other at point A. Prove that :
   (i) point A is equidistant from all the three sides of the triangle.
   (ii) AM bisects angle LMN.

10. Use ruler and compasses only for this question
   (i) Construct Δ ABC, where AB = 3·5 cm, BC = 6 cm and ∠ABC = 60°.
   (ii) Construct the locus of points inside the triangle which are equidistant from BA and BC.
   (iii) Construct the locus of points inside the triangle which are equidistant from B and C.
   (iv) Mark the point P which is equidistant from AB, BC and also equidistant from B and C. Measure and record the length of PB.
   [2010]

11. The given figure shows a triangle ABC in which AD bisects angle BAC. EG is perpendicular bisector of side AB which intersects AD at point F.
   Prove that :



   (i) F is equidistant from A and B.
   (ii) F is equidistant from AB and AC.

12. The bisectors of ∠B and ∠C of a quadrilateral ABCD intersect each other at point P. Show that P is equidistant from the opposite sides AB and CD.

Locus Constructions :

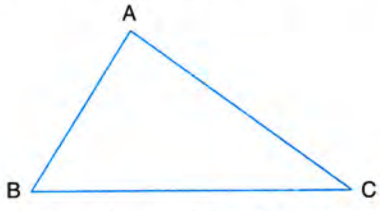
13. Draw a line AB = 6 cm. Draw the locus of all the points which are equidistant from A and B.

14. Draw an angle ABC = 75°. Draw the locus of all the points equidistant from AB and BC.

15. Draw an ∠ABC = 60°, having AB = 4·6 cm and BC = 5 cm. Find a point P equidistant from AB and BC; and also equidistant from A and B.

16. In the figure given below, find a point P on CD equidistant from points A and B.

17. In the given triangle ABC, find a point P equidistant from AB and AC; and also equidistant from B and C.



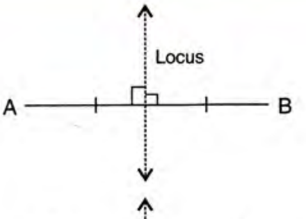
18. Construct a triangle ABC, with AB = 7 cm, BC = 8 cm and ∠ABC = 60°. Locate by construction the point P such that :
   (i) P is equidistant from B and C.
   (ii) P is equidistant from AB and BC.
   Measure and record the length of PB.
   [2000]

19. On a graph paper, draw the lines $x = 3$ and $y = -5$. Now, on the same graph paper, draw the locus of the point which is equidistant from the given lines.

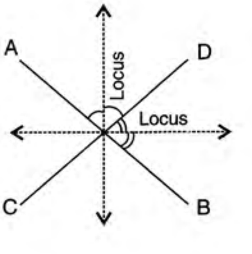
20. On a graph paper, draw the line $x = 6$. Now, on the same graph paper, draw the locus of the point which moves in such a way that its distance from the given line is always equal to 3 units.

16.5 Summary :

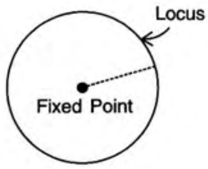
1. The locus of a point, which is equidistant from two fixed points, is the perpendicular bisector of the line segment joining the two fixed points.



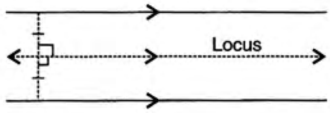
2. The locus of a point, which is equidistant from two intersecting straight lines, consists of a pair of straight lines which bisect the angles between the given lines.



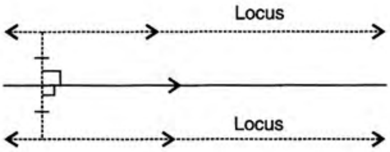
3. The locus of a point, in a plane and at a fixed distance from a given fixed point, is the circumference of the circle with the given fixed point as centre and given fixed distance as radius.



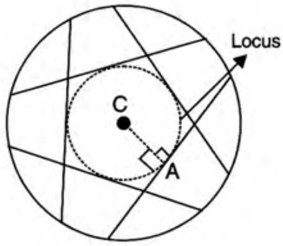
4. The locus of a point, which remains equidistant from two given parallel lines, is a line parallel to the given lines and midway between them.



5. The locus of a point, which is at a given distance from a given line, is a pair of lines parallel to the given line and at the given distance from it.



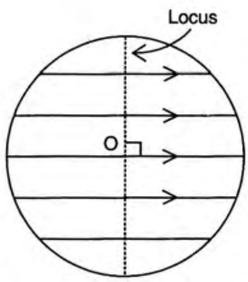
6. The locus of the mid-points of all equal chords in a circle is the circumference of the circle concentric with the given circle and having radius equal to the distance of equal chords from the centre.



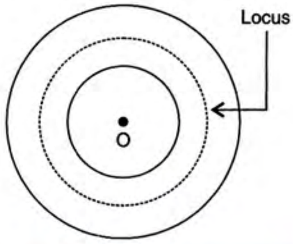
Remember :

Equal chords of a circle are equidistant from its centre.

7. The locus of mid-points of all parallel chords in a circle is the diameter of the circle which is perpendicular to the given parallel chords.



8. The locus of a point equidistant from two concentric circles is the circumference of the circle concentric with the given circles and midway between them.



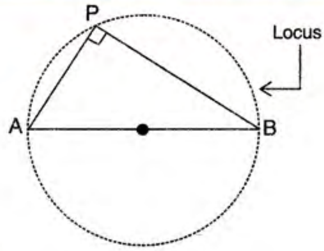
The locus problems, concerning circles, should be attempted after completing the chapter on **circles.**

Remember :

1. To describe the locus of a moving point, state the kind of geometrical figure obtained and its position.
2. Every point satisfying the given condition(s) lies on the locus.
3. Every point on the locus satisfies the given condition(s).
4. The locus can be a straight or a curved line (or lines).

7 A and B are two fixed points. Draw the locus of a point P such that angle APB = 90°.

Solution :



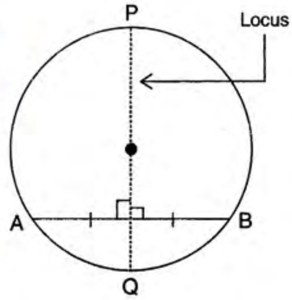
Join AB and draw a circle with AB as diameter. ***The locus of point P is the circumference of this circle.***

Reason :

Since, the angle of semi circle is 90°.

∴ Whatever be the position of point P, the measure of angle APB will always be 90°.

8 AB is a chord of a circle. Draw the locus of a point in the circle so that it is equidistant from A and B.



Solution :

Draw PQ, the perpendicular bisector of chord AB. ***PQ is the required locus, which is a diameter of the circle.***

Reason :

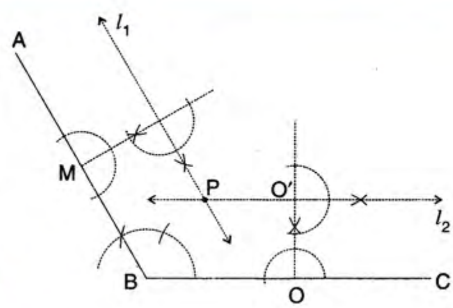
We know each point of the perpendicular bisector of AB is equidistant from A and B. Also, the perpendicular bisector of a chord always passes through the centre of the circle and any chord passing through the centre of the circle is its diameter. Hence, *PQ is the diameter of the circle.*

9 Draw an angle ABC = 120°. Find a point P such that P is at a distance of 3 cm from AB and 2 cm from BC.

Solution :

Construct ∠ABC = 120°. Draw a line l_1, which is parallel to AB and is at a distance of 3 cm from it. Also, draw a line l_2, which is parallel to BC and is at a distance of 2 cm from it.

The two lines l_1 and l_2 intersect at point P. Therefore, **P is the required point.** **Ans.**



EXERCISE 16(B)

Describe the locus for questions 1 to 13 given below :

1. The locus of a point at a distance 3 cm from a fixed point.
2. The locus of points at a distance 2 cm from a fixed line.
3. The locus of the centre of a wheel of a bicycle going straight along a level road.
4. The locus of the moving end of the minute hand of a clock.
5. The locus of a stone dropped from the top of a tower.

6. The locus of a runner running around a circular track and always keeping a distance of 1·5 m from the inner edge.
7. The locus of the door-handle, as the door opens.
8. The locus of points inside a circle and equidistant from two fixed points on the circumference of the circle.
9. The locus of the centres of all circles passing through two fixed points.
10. The locus of vertices of all isosceles triangles having a common base.
11. The locus of a point in space, which is always at a distance of 4 cm from a fixed point.
12. The locus of a point P, so that :

    $AB^2 = AP^2 + BP^2$,

    where A and B are two fixed points.
13. The locus of a point in rhombus ABCD, so that it is equidistant from

    (i) AB and BC; (ii) B and D.
14. The speed of sound is 332 metres per second. A gun is fired. Describe the locus of all the people on the earth's surface, who hear the sound exactly after one second ?
15. Describe :

    (i) The locus of points at distances less than 3 cm from a given point.

    (ii) The locus of points at distances greater than 4 cm from a given point.

    (iii) The locus of points at distances less than or equal to 2·5 cm from a given point.

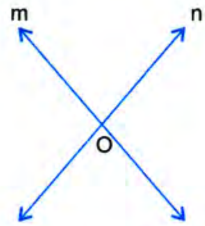
    (iv) The locus of points at distances greater than or equal to 35 mm from a given point.

    (v) The locus of the centre of a given circle which rolls around the outside of a second circle and is always touching it.

    (vi) The locus of the centres of all circles that are tangent to both the arms of a given angle.

    (vii) The locus of the mid-points of all chords parallel to a given chord of a circle.

    (viii) The locus of points within a circle that are equidistant from the end points of a given chord.
16. Sketch and describe the locus of the vertices of all triangles with a given base and a given altitude.
17. In the given figure, obtain all the points equidistant from lines m and n; and 2·5 cm from O.



18. By actual drawing, obtain the points equidistant from lines m and n; and 6 cm from a point P, where P is 2 cm above m, m is parallel to n and m is 6 cm above n.
19. A straight line AB is 8 cm long. Draw and describe the locus of a point which is :

    (i) always 4 cm from the line AB.

    (ii) equidistant from A and B.

    Mark the two points X and Y, which are 4 cm from AB and equidistant from A and B. Describe the figure AXBY. **[2008]**
20. Angle ABC = 60° and BA = BC = 8 cm. The mid-points of BA and BC are M and N respectively. Draw and describe the locus of a point which is :

    (i) equidistant from BA and BC.

    (ii) 4 cm from M.

    (iii) 4 cm from N.

    Mark the point P, which is 4 cm from both M and N, and equidistant from BA and BC. Join MP and NP, and describe the figure BMPN.
21. Draw a triangle ABC in which AB = 6 cm, BC = 4·5 cm and AC = 5 cm. Draw and label :

    (i) the locus of the centres of all circles which touch AB and AC,

    (ii) the locus of the centres of all the circles of radius 2 cm which touch AB.

    Hence, construct the circle of radius 2 cm which touches AB and AC.
22. Construct a triangle ABC, having given AB = 4·8 cm, AC = 4 cm, and ∠A = 75°. Find a point P

    (i) inside the triangle ABC;

    (ii) outside the triangle ABC

    equidistant from B and C; and at a distance of 1·2 cm from BC.
23. O is a fixed point. Point P moves along a fixed line AB. Q is a point on OP produced

such that OP = PQ. Prove that the locus of point Q is a line parallel to AB.

24. Draw an angle ABC = 75°. Find a point P such that P is at a distance of 2 cm from AB and 1·5 cm from BC.

25. Construct a triangle ABC, with AB = 5·6 cm, AC = BC = 9·2 cm. Find the points equidistant from AB and AC; and also 2 cm from BC. Measure the distance between the two points obtained.

26. Construct a triangle ABC, with AB = 6 cm, AC = BC = 9 cm. Find a point 4 cm from A and equidistant from B and C.

27. Ruler and compasses may be used in this question. All construction lines and arcs must be clearly shown and be of sufficient length and clarity to permit assessment.
   (i) Construct a Δ ABC, in which BC = 6 cm, AB = 9 cm and angle ABC = 60°.
   (ii) Construct the locus of all points inside triangle ABC, which are equidistant from B and C.
   (iii) Construct the locus of the vertices of the triangles with BC as base and which are equal in area to triangle ABC.
   (iv) Mark the point Q, in your construction, which would make Δ QBC equal in area to Δ ABC, and isosceles.
   (v) Measure and record the length of CQ.

28. State the locus of a point in a rhombus ABCD, which is equidistant
   (i) from AB and AD;
   (ii) from the vertices A and C.

29. Use graph paper for this question. Take 2 cm = 1 unit on both the axes.
   (i) Plot the points A (1, 1), B (5, 3) and C (2, 7).
   (ii) Construct the locus of points equidistant from A and B.
   (iii) Construct the locus of points equidistant from AB and AC.
   (iv) Locate the point P such that PA = PB and P is equidistant from AB and AC.
   (v) Measure and record the length PA in cm.

30. Construct an isosceles triangle ABC such that AB = 6 cm, BC = AC = 4 cm. Bisect ∠C internally and mark a point P on this bisector such that CP = 5 cm. Find the points Q and R which are 5 cm from P and also 5 cm from the line AB. **[2001]**

31. Use ruler and compasses only for this question. Draw a circle of radius 4 cm and mark two chords AB and AC of the circle of lengths 6 cm and 5 cm respectively.
   (i) Construct the locus of points, inside the circle, that are equidistant from A and C. Prove your construction.
   (ii) Construct the locus of points, inside the circle, that are equidistant from AB and AC.

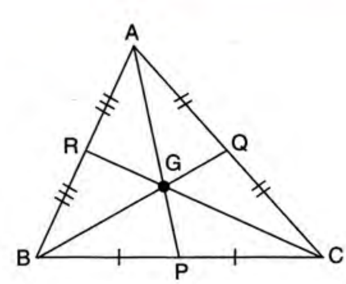
32. Plot the points A(2, 9), B(–1, 3) and C(6, 3) on a graph paper. On the same graph paper, draw the locus of point A so that the area of Δ ABC remains the same as A moves.

33. Construct a triangle BCP given BC = 5 cm, BP = 4 cm and ∠PBC = 45°.
   (i) Complete the rectangle ABCD such that :
      (a) P is equidistant from AB and BC.
      (b) P is equidistant from C and D.
   (ii) Measure and record the length of AB. **[2007]**

16.6 Important :

1. Each triangle has *three medians* which intersect each other at one point only. This point of intersection is called **centroid** of the triangle.

In the figure alongside AP, BQ and CR are the medians of triangle ABC and are intersecting at point G.

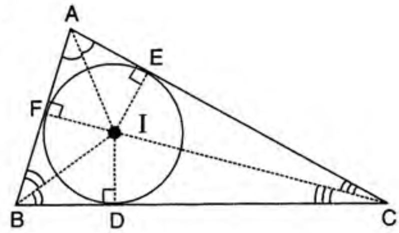


∴ **G** is the **centroid** of triangle ABC.

Centroid of a Δ always divides each median in the ratio 2 : 1

i.e. AG : GP = 2 : 1; BG : GQ = 2 : 1 and CG : GR = 2 : 1.

2. In each triangle, the bisectors of the interior angles meet at a point. This point of intersection is called **incentre** of the triangle and is equidistant from the sides of the triangle.

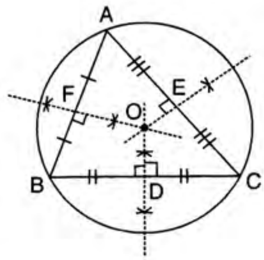


In the figure alongside, AI, BI and CI are the bisectors of angles at A, B and C respectively.

∴ I is the **incentre** of the triangle ABC.

Clearly : I is equidistant from the sides of the triangle, *i.e.* ID = IE = IF = radius of the incircle.

3. The perpendicular bisectors of the three sides of a triangle are concurrent (*i.e.* they intersect each other at the same point). This point of intersection is called **circumcentre** of the triangle and is equidistant from the vertices of the triangle.

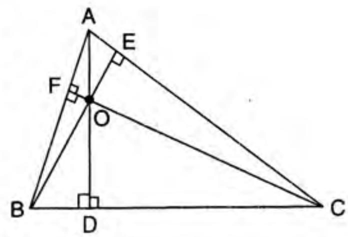


In the figure, DO, EO and FO are the perpendicular bisectors of the sides BC, CA and AB respectively.

∴ O is the **circumcentre** of the triangle and is equidistant from A, B and C.

i.e. OA = OB = OC = radius of the circumcircle.

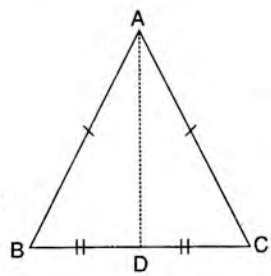
4. Perpendiculars drawn from the vertices of a triangle to the opposite sides (*i.e.* altitudes), are concurrent and the point of concurrency is called **orthocentre** of the triangle.



In the figure, AD, BE and CF are the altitudes corresponding to the sides BC, CA and AB respectively.

∴ Point O is the **orthocentre** of the triangle ABC.

5. In case of an isosceles triangle ABC, if AD is median, then it is bisector of angle A, perpendicular bisector of BC and altitude corresponding to BC

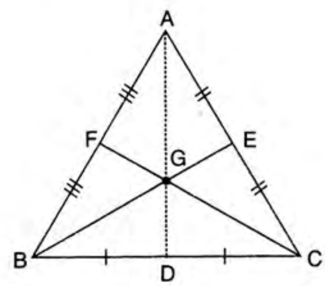


i.e. median AD = bisector of angle A

= perpendicular bisector of opposite side BC

= altitude corresponding to BC.

6. If ABC is an equilateral triangle, then



median AD = bisector of angle A

= perpendicular bisector of BC

= altitude corresponding to BC

median BE = bisector of angle B

= perpendicular bisector of AC

= altitude corresponding to AC

and the same is true for median CF also. Again, if G is centroid of equilateral triangle ABC then,

G = centroid of the Δ ABC = its incentre = its circumcentre = its orthocentre.